

BOUNDARY VALUE PROBLEM OF A NON-STATIONARY STOKES SYSTEM IN A BOUNDED SMOOTH CYLINDER

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ABSTRACT. In this paper, we intend to study the boundary value problem of the non-stationary Stokes system in a bounded smooth cylinder $\Omega \times (0, T)$. As a first step, we consider the problem in half-plane cylinder $\mathbb{R}_+^n \times (0, T)$, $0 < T \leq \infty$. We extend the result of Solonnikov[18] to data in weaker function spaces than the one considered in [18].

1. INTRODUCTION

Let Ω be a n -dimensional bounded C^2 domain for $n \geq 3$. Let us consider the boundary value problem of a non-stationary Stokes system—in other words, a non-stationary linearized system of Navier-Stokes equations—in a cylindrical domain $\Omega \times (0, T)$. For a given initial data $f = (f_1, \dots, f_n)$ and a given boundary data $g = (g_1, \dots, g_n)$, find a unknown vector field $u = (u_1, \dots, u_n)$ and a unknown function p satisfying the system of equations

$$\begin{aligned} u_t - \Delta u + \nabla p &= 0 && \text{in } \Omega \times (0, T), \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0, T), \\ u|_{t=0} &= f && \text{in } \Omega, \\ u|_{\partial\Omega \times (0, T)} &= g && \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{1.1}$$

Here, $\nabla p = (\frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n})$, $\operatorname{div} u = \sum_{1 \leq j \leq n} \frac{\partial u_j}{\partial x_j}$ and Δ is the Laplacian.

The system (1.1) has been studied by many mathematicians. Among them, Solonnikov[18] showed that the system (1.1) has a unique solution satisfying

$$\begin{aligned} &\|u\|_{W_p^{2,1}(\Omega \times (0, T))} \\ &\leq c(T) \left(\|f\|_{\mathcal{B}_p^{2-\frac{2}{p}}(\Omega)} + \|g\|_{\mathcal{B}_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(\partial\Omega \times (0, T))} + \|g_\nu\|_{\mathcal{B}_p^{2-\frac{1}{p}, 1}(\partial\Omega \times (0, T))} \right). \end{aligned} \tag{1.2}$$

Here, g_ν denotes the component of g in the direction of the unit outward normal vector ν . The anisotropic Besov spaces $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\partial\Omega \times (0, T))$, $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha + \frac{1}{2p}}(\partial\Omega \times (0, T))$, and other function spaces are introduced in section 2. By making use of the estimates of the Green matrix in \mathbb{R}_+^n , it is shown that the solution of the non-stationary Stokes system (1.1) in $\mathbb{R}_+^n \times (0, \infty)$ satisfies the estimate

$$\begin{aligned} &\|D_x^2 u\|_{L^p(\mathbb{R}_+ \times (0, \infty))} + \|D_t u\|_{L^p(\mathbb{R}_+ \times (0, \infty))} \\ &\leq c \left(\|f\|_{\mathcal{B}_p^{2-\frac{2}{p}}(\mathbb{R}_+^n)} + \|g\|_{\mathcal{B}_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{\mathcal{B}_p^{2-\frac{1}{p}, 1}(\mathbb{R}^{n-1} \times (0, \infty))} \right), \end{aligned}$$

and then, using flatness near the boundary, the estimate (1.2) has been obtained for the solution of the Stokes system (1.1) in a smooth bounded cylinder $\Omega \times (0, T)$.

The aim of this paper is to extend the result of Solonnikov[18] to data in Besov spaces $(f, g) \in B_p^{\alpha-\frac{1}{p}}(\Omega) \times B_p^{\alpha, \frac{\alpha}{2}}(\partial\Omega \times (0, T))$, $0 < \alpha < 1$, which is weaker than $\mathcal{B}_p^{2-\frac{2}{p}}(\Omega) \times \mathcal{B}_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(\partial\Omega \times (0, T))$.

The system (1.1) can be decomposed into the following two systems:

$$\begin{aligned} v_t - \Delta v + \nabla \pi &= 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} v &= 0 & \text{in } \Omega \times (0, T), \\ v|_{t=0} &= f & \text{in } \Omega, \\ v|_{\partial\Omega \times (0, T)} &= 0 & \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} u_t - \Delta u + \nabla p &= 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} u &= 0 & \text{in } \Omega \times (0, T), \\ u|_{t=0} &= 0 & \text{in } \Omega, \\ u|_{\partial\Omega \times (0, T)} &= g & \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{1.4}$$

The system (1.3) has been studied in various function spaces as a basis of the study of Navier-Stokes system. Indeed, there are many results on an initial boundary value problem of a non-stationary Stokes system with no slip boundary condition (see [1], [11], [16], [20] and references therein).

In this paper, we consider the solvability of (1.4). The solvability of Stokes system (1.4) can lead directly to the solvability of the original Stokes system (1.1). Our result on the system (1.4) could be applied for the study of a boundary value problem of a non-stationary Navier-Stokes system with non-zero boundary data. The following is our main result.

Theorem 1.1. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let $g = (g', g_n) = (g_1, \dots, g_{n-1}, g_n) \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))$ with $g_n \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha + \frac{1}{2p}}(\mathbb{R}^{n-1} \times (0, T))$. Let us also assume $g(x', 0) = 0$ for $x' \in \mathbb{R}^{n-1}$ when $\alpha > \frac{2}{p}$. Then there is a solution of the Stokes system (1.4), Ω is replaced by \mathbb{R}_+^n , with boundary data g such that*

$$\|u\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}_+^n \times (0, T))} \leq c(T) \left(\|g\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0, T))} + \|g_n\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}+\frac{1}{2p}}(\mathbb{R}^{n-1} \times (0, T))} \right). \tag{1.5}$$

This result is optimal in the sense that the restrictions over $\mathbb{R}^{n-1} \times \mathbb{R}$ of the functions contained in $\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}^n \times \mathbb{R})$ are in $\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times \mathbb{R})$ and $\|u\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|u\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}^n \times \mathbb{R})}$ (see [6]).

To obtain estimates of solutions of system (1.4) in a smooth bounded cylinder $\Omega \times (0, T)$, we flatten the boundary and make use of estimates for the solutions of the system (1.4) in $\mathbb{R}_+^n \times (0, T)$. Then we can extend Theorem 1.1 to the Stokes flow in any smooth bounded cylinder.

Corollary 1.2. *Let Ω be a bounded C^2 -domain in \mathbb{R}^n , $1 < p < \infty$, and $0 < \alpha < 1$. Let $g \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\partial\Omega \times (0, T))$ with $g_\nu \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha + \frac{1}{2p}}(\partial\Omega \times (0, T))$ and $\int_{\partial\Omega} g_\nu(P, t) dP = 0$ for all $0 < t < T$. Let us also assume $g(P, 0) = 0$ for $P \in \partial\Omega$ when $\alpha > \frac{2}{p}$. Then there is a solution of (1.4) with boundary data g such that*

$$\|u\|_{\mathcal{B}_p^{\alpha + \frac{1}{p}, \frac{1}{2}\alpha + \frac{1}{2p}}(\Omega \times (0, T))} \leq c(T) \left(\|g\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\partial\Omega \times (0, T))} + \|g_\nu\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2} + \frac{1}{2p}}(\partial\Omega \times (0, T))} \right). \quad (1.6)$$

S. Hofmann, K. Nyström[12], and Z. Shen[17] have also considered Stokes system (1.4) in $\mathbb{R}_+^n \times \mathbb{R}$ and bounded Lipschitz cylinder $\Omega \times (0, T)$, respectively. By single layer and double layer potentials of the Stokes system, S. Hofmann and K. Nyström[12] showed

$$\begin{aligned} \int_{\mathbb{R}_+^n \times \mathbb{R}} x_n |D_x u|^2 dx dt &\leq c \|g\|_{L^2(\mathbb{R}^{n-1} \times \mathbb{R})}^2, \\ \int_{\mathbb{R}_+^n \times \mathbb{R}} \left(x_n |D_x^2 u|^2 + x_n |D_t u|^2 \right) dx dt &\leq c \left(\|g\|_{\mathcal{B}_2^{1, \frac{1}{2}}(\mathbb{R}^{n-1} \times \mathbb{R})}^2 + \int_{\mathbb{R}} \left\| \left\langle \frac{\partial g}{\partial t}, \mathbf{n} \right\rangle \right\|_{\mathcal{B}_2^{-1}(\mathbb{R}^{n-1})}^2 \right). \end{aligned}$$

Their results are compared with our results, that is, theorem 6.1 in section 6.

When Ω is bounded Lipschitz cylinder in \mathbb{R}^{n+1} , Z. Shen[17] showed

$$\|u^*\|_{L^2(\Omega \times (0, T))} \leq c(T) \|g\|_{L^2(\partial\Omega \times (0, T))} \quad (1.7)$$

and

$$\begin{aligned} &\|(D_x u)^*\|_{L^2(\partial\Omega \times (0, T))} + \|(D_t^{\frac{1}{2}} u)^*\|_{L^2(\partial\Omega \times (0, T))} + \|u^*\|_{L^2(\partial\Omega \times (0, T))} \\ &\leq c(T) \left(\|g\|_{\mathcal{B}_2^{1, \frac{1}{2}}(\partial\Omega \times (0, T))} + \left(\int_0^T \left\| \left\langle \frac{\partial g}{\partial t}, \mathbf{n} \right\rangle \right\|_{\mathcal{B}_2^{-1}(\partial\Omega)}^2 \right)^{\frac{1}{2}} \right). \end{aligned} \quad (1.8)$$

Here u^* denotes the non-tangential limit of u .

We are not sure that the solutions satisfying (1.7) and (1.8) are in Besov spaces $\mathcal{B}_2^{\frac{1}{2}, \frac{1}{4}}(\Omega \times (0, T))$ and $\mathcal{B}_2^{\frac{3}{2}, \frac{3}{4}}(\Omega \times (0, T))$, respectively, since for the non-stationary Stokes system it is still open problem whether the L^2 norm of a non-tangential limit of the solution is equivalent to an area integral of the solution.

On the other hand, for the solutions of a elliptic equation, a parabolic equation, and the stationary Stokes system it is well known that the L^2 norms of a non-tangential limit of the solutions are equivalent to the area integrals of the solutions (see [8], [9] on the elliptic equation, see [2] on the parabolic equation, and see [5] for the stationary Stokes system).

We organized the paper in the following way. In section 2, we introduce the anisotropic function spaces. In section 3, we see that without loss of generality we can assume the boundary data $g_n = 0$ and then represent the solution of stokes system in $\mathbb{R}_+^n \times (0, \infty)$ by some integral formula. In section 4 we study the embedding properties of the functions in weighted Sobolev spaces into anisotropic spaces and we introduce on the atomic decomposition of the functions in anisotropic spaces. In section 5, we derive pointwise estimates of the solution of the Stokes system (1.4) in $\mathbb{R}_+^n \times (0, \infty)$ when the boundary data $g = (g', 0)$ is given by an atom. In section 6, we show that the solution of the Stokes system (1.4) in $\mathbb{R}_+^n \times (0, \infty)$ is in some weighted Sobolev spaces in $\mathbb{R}_+^n \times (0, \infty)$ using the estimates of section 5 when the boundary data is in the proper anisotropic space. In section 7 we derive the estimates as in Theorem 1.1 for the boundary data $g = (g', 0)$. Theorem 1.1 for any boundary data $g = (g', g_n)$ will be proved combining the result of Theorem 7.1 in section 7 and Proposition 3.1 in section 3.

2. BESOV SPACES AND ANISOTROPIC BESOV SPACES

We denote $x = (x', x_n) \in \mathbb{R}_+^n$ for $x' \in \mathbb{R}^{n-1}$ and denote $D_{x_n}^{l_0} D_x^{k_0} D_t^{m_0} = \frac{\partial^{l_0}}{\partial x_n} \frac{\partial^{|k_0|}}{\partial x^{k_0}} \frac{\partial^{m_0}}{\partial t}$ for multi index l_0, k_0, m_0 . Throughout this paper we denote by c various generic constants and by $c(*, \dots, *)$ the constants depending only on the quantities appearing in the subindex.

Let Ω be \mathbb{R}_+^n or a bounded domain. For $1 \leq p \leq \infty$ and $0 < \alpha$, the Besov space $\mathcal{B}_p^\alpha(\Omega)$ is set of functions satisfying

$$\begin{aligned} \|f\|_{\mathcal{B}_p^\alpha(\Omega)}^p &= \|f\|_{W_p^{[\alpha]}(\Omega)}^p + \sum_{|k|=[\alpha]} \int_{\Omega} \int_{\Omega} \frac{|D_x^k f(x) - D_y^k f(y)|^p}{|x - y|^{n+p(\alpha-|k|)}} dx dy < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{\mathcal{B}_\infty^\alpha(\Omega)} &= \sup_{x \in \Omega} |D_x^{[\alpha]} u(x)| + \sup_{|k|=[\alpha]} \sum_{x, y \in \Omega} \frac{|D_x^k u(x) - D_y^k u(y)|}{|x - y|^{\alpha-|k|}} < \infty, \quad p = \infty. \end{aligned}$$

Here $[\alpha]$ denotes the largest integer less than α and $W_p^{[\alpha]}(\Omega)$ is the usual Sobolev space in Ω .

For interval such as $I = (0, T)$, $(0, \infty)$ or \mathbb{R} , $\mathcal{B}_p^\alpha(I)$ is defined similarly for $1 \leq p \leq \infty$ and $\alpha > 0$.

For $0 < \alpha < 2$ and $0 < \beta < 2$, we define the anisotropic Besov spaces $\mathcal{B}_p^{\alpha, \beta}(\Omega \times I)$ by the Banach spaces

$$\mathcal{B}_p^{\alpha, \beta}(\Omega \times I) = L^p(I; \mathcal{B}_p^\alpha(\Omega)) \cap L^p(\Omega; \mathcal{B}_p^\beta(I)) \quad (2.1)$$

with norm

$$\begin{aligned}\|u\|_{\mathcal{B}_p^{\alpha,\beta}(\Omega \times I)}^p &:= \int_I \|u(\cdot, t)\|_{\mathcal{B}_p^\alpha(\Omega)}^p dt + \int_\Omega \|u(x, \cdot)\|_{\mathcal{B}_p^\beta(I)}^p dx \text{ for } 1 \leq p < \infty, \\ \|u\|_{\mathcal{B}_\infty^{\alpha,\beta}(\Omega \times I)} &:= \sup_{t \in I} \|u(\cdot, t)\|_{\mathcal{B}_\infty^\alpha(\Omega)} + \sup_{x \in \Omega} \|u(x, \cdot)\|_{\mathcal{B}_\infty^\beta(I)}.\end{aligned}$$

It is well known theory that the usual Besov spaces are real interpolation spaces of Sobolev spaces:

$$\begin{aligned}\mathcal{B}_p^\alpha(\Omega) &= \begin{cases} (W_p^1(\Omega), L^p(\Omega))_{1-\alpha,p} & \text{if } 0 < \alpha < 1, \\ (W_p^2(\Omega), W_p^1(\Omega))_{2-\alpha,p} & \text{if } 1 < \alpha < 2. \end{cases} \\ \mathcal{B}_p^{\frac{1}{2}\alpha}(I) &= (W_p^1(I), L^p(I))_{1-\frac{1}{2}\alpha,p} \text{ if } 0 < \alpha < 2,\end{aligned}$$

for $1 \leq p \leq \infty$ (see Proposition 2.17 in [14]). It is also well-known that for $1 \leq p < \infty$

$$\begin{aligned}L^p(I; \mathcal{B}_p^\alpha(\Omega)) &= \begin{cases} (L^p(I; W_p^1(\Omega)), L^p(I; L^p(\Omega)))_{1-\alpha,p} & \text{if } 0 < \alpha < 1, \\ (L^p(I; W_p^2(\Omega)), L^p(I; W_p^1(\Omega)))_{2-\alpha,p} & \text{if } 1 < \alpha < 2 \end{cases} \\ L^p(\Omega; \mathcal{B}_p^{\frac{1}{2}\alpha}(I)) &= (L^p(\Omega; W_p^1(I)), L^p(\Omega; L^p(I)))_{1-\frac{1}{2}\alpha,p}\end{aligned} \quad (2.2)$$

(see Comment 5.8.6 in [3]).

3. SOLUTION FORMULA OF THE STOKES SYSTEM IN $\mathbb{R}_+^n \times (0, \infty)$

Let $g \in C_c^\infty(\mathbb{R}^{n-1} \times (0, \infty))$. We decompose g by $g = g^1 + g^2 = (Rg_n, g_n) + (g' - Rg_n, 0)$, where $R = (R_1, R_2, \dots, R_{n-1})$ is Riesz transform. Let

$$\phi(x, t) = -\omega_n \int_{\mathbb{R}^{n-1}} E(x' - y', x_n) g_n(y', t) dy',$$

where E is a fundamental solution of Laplace equation. Then, $(\nabla \phi, -\phi_t)$ satisfies the Stokes system (1.4) for $\Omega = \mathbb{R}_+^n$ with boundary data g^1 . The following estimate is well known property of the singular integral operator (see [19]).

Proposition 3.1. *Let $1 < p < \infty$ and $\alpha > 0$. Then there is a positive constant c such that*

$$\|\nabla \phi\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{\alpha}{2}+\frac{1}{2p}}(\mathbb{R}_+^n \times (0, T))} \leq c(T) \|g_n\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}+\frac{1}{2p}}(\mathbb{R}^{n-1} \times (0, T))}$$

for $0 < T < \infty$.

Let (u, p) be the solution of the Stokes system (1.4) in $\Omega = \mathbb{R}_+^n$ with boundary data g^2 . Then, $(\nabla \phi + u, -\phi_t + p)$ satisfies the Stokes system (1.4) with boundary data g . Hence, to prove theorem 1.1 we have only to consider the Stokes system (1.4) with the boundary data g^2 . Note that $g_n^2 = 0$.

From now on, without loss of generality, we assume $g_n = 0$. The solution (u, p) of the Stokes system (1.4) with boundary data g with $g_n = 0$ is represented by

$$\begin{aligned} u^i(x, t) &= \sum_{j=1}^{n-1} \int_0^t \int_{\mathbb{R}^{n-1}} K_{i,j}(x' - y', x_n, t - s) g_j(y', s) dy' ds, \\ p(x, t) &= \sum_{j=1}^{n-1} \int_0^t \int_{\mathbb{R}^{n-1}} \pi_j(x' - y', x_n, t - s) g_j(y', s) dy' ds, \end{aligned} \quad (3.1)$$

where

$$K_{ij}(x, t) = -2\delta_{ij} D_{x_n} \Gamma(x, t) + 4G_{ij}(x, t)$$

and

$$\pi_j(x, t) = -2\delta(t) \frac{\partial^2}{\partial x_j \partial x_n} E(x) + 4 \frac{\partial^2}{\partial x_n^2} A(x, t) + 4 \frac{\partial}{\partial t} \frac{\partial}{\partial x_j} A(x, t),$$

where Γ and E are the fundamental solutions of the heat equation and Laplace equation, respectively, and

$$\begin{aligned} G_{ij}(x, t) &= D_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} D_{z_n} \Gamma(z, t) D_{x_i} E(x - z) dz, \\ A(x, t) &= \int_{\mathbb{R}^{n-1}} \Gamma(z', 0, t) E(x' - z', x_n) dz'. \end{aligned}$$

G_{ij} and A satisfy the estimates

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} G_{ij}(x, t)| \leq \frac{c}{t^{m_0 + \frac{1}{2}} (|x|^2 + t)^{\frac{1}{2}n + \frac{1}{2}k_0} (x_n^2 + t)^{\frac{1}{2}l_0}}, \quad (3.2)$$

$$|D_x^j D_t^m A(x, t)| \leq \frac{c}{t^{m + \frac{1}{2}} (|x|^2 + t)^{\frac{n-2+|j|}{2}}}, \quad (3.3)$$

where $1 \leq i \leq n$ and $1 \leq j \leq n-1$ (see [15] and [18]). The estimates (3.2) of G_{ij} and the estimate of Gaussian kernel Γ imply that

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} K_{ij}(x, t)| \leq \frac{c}{t^{m_0 + \frac{1}{2}} (|x|^2 + t)^{\frac{1}{2}n + \frac{1}{2}k_0} (x_n^2 + t)^{\frac{1}{2}l_0}}. \quad (3.4)$$

4. PRELIMINARY THEORIES

4.1. Estimates weighted Sobolev spaces in Anisotropic spaces.

Lemma 4.1. *Let $0 < \alpha < 1$ and $1 \leq p < \infty$. Let $u \in C^\infty(\mathbb{R}_+^n \times (0, \infty))$. Then*

$$\begin{aligned} \|u\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}_+^n \times (0, \infty))}^p &\leq c \int \int_{\mathbb{R}_+^n \times (0, \infty)} (x_n \wedge t^{\frac{1}{2}})^{p-p\alpha} \left(|D_x u|^p + |u|^p \right) \\ &\quad + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha} \left(|u|^p + |D_t u|^p \right) dx dt. \end{aligned} \quad (4.1)$$

Here, $a \wedge b = \min\{a, b\}$.

Proof. By the property of real interpolation (2.2)₁, we note that

$$\begin{aligned} &\|u\|_{L^p((0, \infty); \mathcal{B}_p^\alpha(\mathbb{R}_+^n))} \\ &\leq \inf \left[\left(\int_0^\infty \|s^{1-\alpha} f(s)\|_{L^p((0, \infty); W_p^1(\mathbb{R}_+^n))}^p s^{-1} ds \right)^{\frac{1}{p}} + \left(\int_0^\infty \|s^{1-\alpha} f'(s)\|_{L^p((0, \infty); L^p(\mathbb{R}_+^n))}^p s^{-1} ds \right)^{\frac{1}{p}} \right], \end{aligned}$$

where infimum is taken by $f : [0, \infty) \rightarrow L^p((0, \infty); W_p^1(\mathbb{R}_+^n)) + L^p((0, \infty); L^p(\mathbb{R}_+^n))$ satisfying $f(0) = u$ (see Theorem 3.12.2 in [3]). Define $f(s) = u(x', x_n + s, t + s^2)$. Then $f(0) = u$, and hence

$$\begin{aligned} \|u\|_{L^p((0, \infty); \mathcal{B}_p^\alpha(\mathbb{R}_+^n))}^p &\leq \int_0^\infty \int_0^\infty \|s^{1-\alpha} f(s)\|_{W_p^1(\mathbb{R}_+^n)}^p s^{-1} dt ds + \int_0^\infty \int_0^\infty \|s^{1-\alpha} f'(s)\|_{L^p(\mathbb{R}_+^n)}^p s^{-1} dt ds \\ &\leq c \int_0^\infty \int_0^\infty \int_{\mathbb{R}_+^n} s^{p-p\alpha-1} \left(|u(x', x_n + s, t + s^2)|^p + |D_x u(x', x_n + s, t + s^2)|^p \right. \\ &\quad \left. + s^p |D_t u(x', x_n + s, t + s^2)|^p \right) dx dt ds. \end{aligned}$$

Changing variables and exchanging the order of integrations, the right hand side of the above last inequality is less than

$$\int_0^\infty \int_{\mathbb{R}_+^n} \left(|u(x, t)|^p + |D_x u(x, t)|^p \right) \int_0^{x_n \wedge t^{\frac{1}{2}}} s^{p-\alpha p-1} ds + |D_t u(x, t)|^p \left(\int_0^{x_n \wedge t^{\frac{1}{2}}} s^{2p-\alpha p-1} ds \right) dx dt.$$

Since $p - p\alpha > 0$, the above is dominated by

$$\int_0^\infty \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{p-p\alpha} \left(|D_x u(x, t)|^p + |u(x, t)|^p + (x_n \wedge t^{\frac{1}{2}})^p |D_t u(x, t)|^p \right) dx dt.$$

Next, we define $g(s) = u(x', x_n + s^{\frac{1}{2}}, t + s)$. Then, $g(0) = u$ and by the property of real interpolation $(2.2)_2$, we have

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}_+^n; \mathcal{B}_p^{\frac{1}{2}\alpha}(0, \infty))}^p &\leq \int_0^\infty \int_{\mathbb{R}_+^n} \left(\|s^{1-\frac{\alpha}{2}} g(s)\|_{W_p^1(0, \infty)}^p s^{-1} ds + \|s^{1-\frac{\alpha}{2}} g'(s)\|_{L^p(0, \infty)}^p s^{-1} \right) ds \\ &\leq \int_0^\infty \int_0^\infty \int_{\mathbb{R}_+^n} s^{p-\frac{\alpha p}{2}-1} \left(|u(x', x_n + s, t + s^2)|^p + |D_t u(x', x_n + s, t + s^2)|^p \right) \\ &\quad + s^{\frac{p}{2}-\frac{\alpha p}{2}-1} |D_{x_n} u(x', x_n + s, t + s^2)|^p dx dt ds. \end{aligned}$$

Changing variables and exchanging the order of integrations, the right hand side of the above last inequality is less than

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}_+^n} \left(|u(x, t)|^p + |D_t u(x, t)|^p \right) \int_0^{x_n^2 \wedge t} s^{p-\frac{\alpha p}{2}-1} ds + |D_{x_n} u(x, t)|^p \left(\int_0^{x_n^2 \wedge t} s^{\frac{p}{2}-\frac{\alpha p}{2}-1} ds \right) dx dt \\ &\leq c \int_0^\infty \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha} \left(|D_t u(x, t)|^p + |u(x, t)|^p + (x_n \wedge t^{\frac{1}{2}})^{-p} |D_{x_n} u(x, t)|^p \right) dx dt. \end{aligned}$$

□

Lemma 4.2. *Let $1 < \alpha < 2$ and $1 \leq p < \infty$. Let $u \in C^\infty(\mathbb{R}_+^n \times (0, \infty))$. Then*

$$\begin{aligned} \|u\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}_+^n \times (0, \infty))}^p &\leq c \int \int_{\mathbb{R}_+^n \times (0, \infty)} (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha} \left(|D_x^2 u|^p + |D_x u|^p + |u|^p + |D_t u|^p \right) \\ &\quad + (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha} \left(|D_t u|^p + |D_x D_t u|^p + |D_x u|^p \right) dx dt. \end{aligned} \quad (4.2)$$

Proof. As in the proof of Lemma 4.1, we define $f(s) = u(x, x_n + s, t + s^2)$. Then $f(0) = u$, and by the property of real interpolation (2.2)₁, we have

$$\begin{aligned}
& \|u\|_{L^p((0,\infty);\mathcal{B}_p^\alpha(\mathbb{R}_+^n))}^p \\
& \leq \int_0^\infty \int_0^\infty \|s^{2-\alpha} f(s)\|_{W_p^2(\mathbb{R}_+^n)}^p s^{-1} dt ds + \int_0^\infty \int_0^\infty \|s^{2-\alpha} f'(s)\|_{W_p^1(\mathbb{R}_+^n)}^p s^{-1} dt ds \\
& \leq \int_0^\infty \int_0^\infty s^{2p-p\alpha-1} \int_{\mathbb{R}_+^n} |u(x', x_n + s, t + s^2)|^p + |D_x u(x', x_n + s, t + s^2)|^p \\
& \quad + |D_x^2 u(x', x_n + s, t + s^2)|^p \\
& \quad + s^p \left(|D_t u(x', x_n + s, t + s^2)|^p + |D_x D_t u(x', x_n + s, t + s^2)|^p \right) dx dt ds.
\end{aligned}$$

By changing variables and exchanging the order of integration, the right hand side of the above last inequality is less than

$$\begin{aligned}
& c \int \int_{\mathbb{R}_+^n \times (0,\infty)} \left(|D_x^2 u|^p + |D_x u|^p + |u|^p \right) \int_0^{x_n \wedge t^{\frac{1}{2}}} s^{2p-p\alpha-1} ds \\
& \quad + \left(|D_t u|^p + |D_x D_t u|^p \right) \left(\int_0^{x_n \wedge t^{\frac{1}{2}}} s^{3p-p\alpha-1} ds \right) dx dt \\
& \leq c \int \int_{\mathbb{R}_+^n \times (0,\infty)} (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha} \left(|D_x^2 u|^p + |D_x u|^p + |u|^p \right) \\
& \quad + (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha} \left(|D_t u|^p + |D_x D_t u|^p \right) dx dt.
\end{aligned}$$

Next, we define $g(s) = u(x', x_n + s^{\frac{1}{2}}, t + s) - s^{\frac{1}{2}} D_{x_n} u(x', x_n + s^{\frac{1}{2}}, t + s)$. Then by the property of real interpolation (2.2)₂, we have

$$\|u\|_{L^p(\mathbb{R}_+^n; \mathcal{B}_p^{\frac{1}{2}\alpha}(0,\infty))}^p \leq \int_0^\infty \int_{\mathbb{R}_+^n} \left(\|s^{1-\frac{\alpha}{2}} g(s)\|_{W_p^1(0,\infty)}^p s^{-1} + \int_0^\infty \|s^{1-\frac{\alpha}{2}} g'(s)\|_{L^p(0,\infty)}^p s^{-1} \right) dx ds. \quad (4.3)$$

The first integration term of the right-hand side of (4.3) is dominated by

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \int_{\mathbb{R}_+^n} s^{p(1-\frac{\alpha}{2})-1} \left(|D_t u(x', x_n + s^{\frac{1}{2}}, t + s)|^p + |u(x', x_n + s^{\frac{1}{2}}, t + s)|^p \right) \\
& \quad + s^{\frac{3p}{2}-\frac{\alpha p}{2}-1} \left(|D_{x_n} D_t u(x', x_n + s^{\frac{1}{2}}, t + s)|^p + |D_{x_n} u(x', x_n + s^{\frac{1}{2}}, t + s)|^p \right) dx dt ds \\
& \leq c \int_0^\infty \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{p(2-\alpha)} \left(|D_t u(x, t)|^p + |u(x, t)|^p \right) \\
& \quad + (x_n \wedge t^{\frac{1}{2}})^{(3-\alpha)p} \left(|D_{x_n} u(x, t)|^p + |D_{x_n} D_t u(x, t)|^p \right) dx dt.
\end{aligned}$$

To estimate the second integration term on the right-hand side of (4.3), we note that $g'(s) = D_t u + \frac{1}{2} D_{x_n}^2 u - s^{\frac{1}{2}} D_{x_n} D_t u$. By the same reasoning as for the estimate of the first

term, the second term is dominated by

$$\int_0^\infty \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{p(2-\alpha)} \left(|D_t(x, t)|^p + |D_x^2 u(x, t)|^p \right) + (x_n \wedge t^{\frac{1}{2}})^{(3-\alpha)p} |D_x D_t u(x, t)| dx dt.$$

□

4.2. Atom decomposition of the functions in the anisotropic Besov spaces. Now, we introduce atomic decomposition of functions in anisotropic space (see [4] for the reference and see also [14] for atomic decomposition of functions in Besov spaces).

Definition 4.3 (Definition 5.2 in [4]). *Let $\alpha > 0$ and $1 \leq p \leq \infty$. An (α, p) -atom is a function in $\mathbb{R}^{n-1} \times \mathbb{R}$ satisfying*

$$|a| \leq r^{\alpha - \frac{n+1}{p}}, \quad |D_{x'} a| \leq r^{\alpha - \frac{n+1}{p} - 1}, \quad |D_t a| \leq r^{\alpha - \frac{n+1}{p} - 2}, \quad \text{supp } a \subset \Delta(y'_0, r) \times (t_0, t_0 + r^2) \quad (4.4)$$

for some $r > 0$ and $(y'_0, t_0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $\Delta(y'_0, r) = \{y' \in \mathbb{R}^{n-1} \mid |y'_0 - y'| < r\}$.

Proposition 4.4 (Theorem 5.10 in [4]). *Let $\alpha > 0$ and $1 \leq p \leq \infty$ and $g \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times \mathbb{R})$. Then there are sequences $\{a_k\}_{1 \leq k < \infty}$ and $\{c_k\}_{1 \leq k < \infty}$ of atoms and real numbers such that $g = \sum c_k a_k$ and*

$$\left(\sum_{1 \leq k < \infty} |c_k|^p \right)^{\frac{1}{p}} \leq c \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times \mathbb{R})}.$$

5. POINTWISE ESTIMATES OF SOLUTION IN $\mathbb{R}_+^n \times (0, \infty)$

In this section, we would like to derive pointwise estimates of solution of the Stokes system (1.4) with boundary data $g = (g', 0) \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times \mathbb{R})$, for $1 \leq p \leq \infty$ and $0 < \alpha < 1$ (by the reasoning in section 3, we may assume $g_n = 0$). From Proposition 4.4, without loss of generality, we assume that the component functions g'_k , $1 \leq k \leq n-1$ of $g = (g', 0)$ consist of (α, p) -atoms. For simplicity, assume $g'_k = a \delta_{kj}$ for fixed $1 \leq j \leq n-1$, where a is an (α, p) -atom such that $\text{supp } a \subset \Delta(0, r) \times (0, r^2)$. By (3.1), the solution $u = (u_1, \dots, u_n)$ of (1.4) is represented by

$$u^i(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} K_{ij}(x' - y', x_n, t - s) a(y', s) dy' ds, \quad 1 \leq i \leq n. \quad (5.1)$$

Lemma 5.1. *Let $t \geq (2r)^2$. Then*

$$\begin{aligned} & |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u(x, t)| \\ & \leq c \begin{cases} r^{\alpha - \frac{n+1}{p} + n+1} t^{-\frac{1}{2} - m_0} (|x'|^2 + x_n^2 + t)^{-\frac{n+k_0}{2}} (x_n^2 + t)^{-\frac{l_0}{2}} & \text{if } |x'| \geq 2r, \\ r^{\alpha - \frac{n+1}{p} + n+1} t^{-\frac{1}{2} - m_0} (x_n^2 + t)^{-\frac{n+k_0+l_0}{2}} & \text{if } |x'| \leq 2r. \end{cases} \end{aligned} \quad (5.2)$$

Proof. Note that for $1 \leq i \leq n$, we have

$$D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u^i(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} K_{ij}(x' - y', x_n, t - s) a(y', s) dy' ds. \quad (5.3)$$

Since $t \geq (2r)^2$, from the estimate (3.4) of K_{ij} and (4.4), we have

$$\begin{aligned} & |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u^i(x, t)| \\ & \leq cr^{\alpha - \frac{n+1}{p}} \int_0^t \int_{|y'| < r} (t-s)^{-\frac{1}{2}-m_0} (|x' - y'|^2 + x_n^2 + t - s)^{-\frac{n}{2} - \frac{k_0}{2}} (x_n^2 + t - s)^{-\frac{l_0}{2}} dy' ds \\ & \leq cr^{\alpha - \frac{n+1}{p}} \int_0^t \int_{|y'| < r} t^{-\frac{1}{2}-m_0} (|x' - y'|^2 + x_n^2 + t)^{-\frac{n}{2} - \frac{k_0}{2}} (x_n^2 + t)^{-\frac{l_0}{2}} dy' ds. \end{aligned} \quad (5.4)$$

Note that for $\theta_1 > \frac{n-1}{2}$, we have

$$\int_{|y'| < r} (|x' - y'|^2 + x_n^2 + t)^{-\theta_1} dy' \leq c \begin{cases} r^{n-1} (|x'|^2 + x_n^2 + t)^{-\theta_1}, & \text{if } |x'| \geq 2r \\ r^{n-1} (x_n^2 + t)^{-\theta_1}, & \text{if } |x'| \leq 2r. \end{cases} \quad (5.5)$$

Taking $\theta_1 = \frac{n+k_0}{2} > \frac{n-1}{2}$ in (5.5) and applying to the right hand side of (5.4), we obtain the estimate (5.2). \square

Lemma 5.2. *Let $t \leq (2r)^2$.*

(1) *If $|x'| \leq 2r$ and $x_n^2 \leq t$, then*

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u(x, t)| \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 2m_0} x_n^{-(k_0+l_0)} & \text{for } k_0 + l_0 \geq 1, \\ cr^{\alpha - \frac{n+1}{p} - 2m_0} \ln(1 + \frac{t}{x_n^2}) & \text{for } k_0 + l_0 = 0. \end{cases} \quad (5.6)$$

(2) *If $|x'| \leq 2r$ and $x_n^2 \geq t$, then*

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u(x, t)| \leq cr^{\alpha - \frac{n+1}{p} - 2m_0} x_n^{-(k_0+l_0+1)} t^{\frac{1}{2}}. \quad (5.7)$$

(3) *If $|x'| \geq 2r$ and $x_n^2 \leq t$, then*

$$\begin{aligned} & |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u(x, t)| \\ & \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 2m_0 + n - 1} (|x'|^2 + x_n^2)^{\frac{-n-k_0}{2}} \ln(1 + \frac{t}{x_n^2}) & \text{for } l_0 = 1, \\ cr^{\alpha - \frac{n+1}{p} - 2m_0 + n - 1} (|x'|^2 + x_n^2)^{\frac{-n-k_0}{2}} (x_n^2 + t)^{\frac{1-l_0}{2}} & \text{for } l_0 \neq 1. \end{cases} \end{aligned} \quad (5.8)$$

(4) *If $|x'| \geq 2r$ and $x_n^2 \geq t$, then*

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u(x, t)| \leq cr^{\alpha - \frac{n+1}{p} - 2m_0 + n - 1} x_n^{-l_0} (|x'|^2 + x_n^2)^{-\frac{n+k_0}{2}} t^{\frac{1}{2}}. \quad (5.9)$$

Proof. By the uniqueness of the solution of the initial-boundary value problem, we have

$D_t u(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} K(x' - y', x_n, t - s) D_s a(y', s) dy' ds$. This implies

$$D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u^i(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n}^{l_0} D_{x'}^{k_0} K_{ij}(x' - y', x_n, t - s) D_s^{m_0} a(y', s) dy' ds. \quad (5.10)$$

Since $t \leq (2r)^2$, from the estimate (3.4) of K_{ij} and (4.4), and taking $\theta_1 := \frac{n+k_0}{2} > \frac{n-1}{2}$ in (5.5), we have

$$\begin{aligned}
& |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u^i(x, t)| \\
& \leq cr^{\alpha - \frac{n+1}{p} - 2m_0} \int_0^t \int_{|y'| < r} (t-s)^{-\frac{1}{2}} (|x' - y'|^2 + x_n^2 + t - s)^{\frac{-n-k_0}{2}} (x_n^2 + t - s)^{-\frac{l_0}{2}} dy' ds \\
& \leq cr^{\alpha - \frac{n+1}{p} - 2m_0} \int_0^t \int_{|y'| < r} s^{-\frac{1}{2}} (|x' - y'|^2 + x_n^2 + s)^{\frac{-n-k_0}{2}} (x_n^2 + s)^{-\frac{l_0}{2}} dy' ds \\
& \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 2m_0 + n-1} (|x'|^2 + x_n^2)^{\frac{-n-k_0}{2}} \int_0^t s^{-\frac{1}{2}} (x_n^2 + s)^{-\frac{l_0}{2}} ds & \text{if } |x'| \geq 2r, \\ cr^{\alpha - \frac{n+1}{p} - 2m_0} \int_0^t s^{-\frac{1}{2}} (x_n^2 + s)^{\frac{-n-k_0-1}{2}} ds & \text{if } |x'| \leq 2r. \end{cases} \quad (5.11)
\end{aligned}$$

Note that for $\theta_2 \geq 0$, if $x_n^2 \geq t$, then

$$\int_0^t s^{-\frac{1}{2}} (x_n^2 + s)^{-\theta_2} ds \leq cx_n^{-2\theta_2} \int_0^t s^{-\frac{1}{2}} ds \leq cx_n^{-2\theta_2} t^{\frac{1}{2}}, \quad (5.12)$$

and if $x_n^2 \leq t$, then

$$\begin{aligned}
\int_0^t s^{-\frac{1}{2}} (x_n^2 + s)^{-\theta_2} ds & \leq x_n^{-2\theta_2} \int_0^{x_n^2} s^{-\frac{1}{2}} ds + \int_{x_n^2}^t s^{-\frac{1}{2}-\theta_2} ds \\
& \leq \begin{cases} ct^{-\theta_2+\frac{1}{2}} & \text{for } \theta_2 < \frac{1}{2}, \\ cx_n^{-2\theta_2+1} & \text{for } \theta_2 > \frac{1}{2}, \\ c \ln(1 + \frac{t}{x_n^2}) & \text{for } \theta_2 = \frac{1}{2}. \end{cases} \quad (5.13)
\end{aligned}$$

Taking $\theta_2 := \frac{k_0+l_0+1}{2} \geq \frac{1}{2}$ in (5.12) - (5.13) and applying to the right hand side of (5.11)₂, we obtain the estimates (5.6) and (5.7). And taking $\theta_2 := \frac{l_0}{2} \geq 0$ in (5.12) - (5.13) and applying to the right hand side of (5.11)₁, we obtain the estimates (5.8) and (5.9). \square

Lemma 5.3. *Let $t \leq (2r)^2$ and $|x'| \leq 2r$. Let $k_0 \geq 1$.*

(1) *For $x_n^2 \leq t$ we have*

$$|D_{x'}^{k_0} D_{x_n}^{l_0} u(x, t)| \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 1} \ln(1 + \frac{t}{x_n^2}) & \text{if } l_0 + k_0 = 1, \\ cr^{\alpha - \frac{n+1}{p} - 1} x_n^{1-k_0-l_0} & \text{if } l_0 + k_0 \geq 2. \end{cases} \quad (5.14)$$

(2) *For $t \leq x_n^2$, we have*

$$|D_{x'}^{k_0} D_{x_n}^{l_0} u(x, t)| \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 1} x_n^{-(k_0+l_0)} t^{\frac{1}{2}} & x_n \leq r, \\ cr^{\alpha - \frac{n+1}{p} + n-2} x_n^{-(n-1+k_0+l_0)} t^{\frac{1}{2}} & x_n \geq r. \end{cases} \quad (5.15)$$

Proof. Since $k_0 \geq 1$, we have

$$\begin{aligned}
|D_{x_n}^{l_0} D_{x'}^{k_0} u^i(x, t)| &= \left| \int_0^t \int_{\mathbb{R}^{n-1}} D_{x'}^{k_0-1} D_{x_n}^{l_0} K_{ij}(x' - y', x_n, t-s) D_{y'} a(y', s) dy' ds \right| \\
&\leq cr^{\alpha - \frac{n+1}{p} - 1} \int_0^t \int_{|x' - y'| \leq 3r} (t-s)^{-\frac{1}{2}} (x_n^2 + t-s)^{-\frac{l_0}{2}} \\
&\quad \times (|x' - y'|^2 + x_n^2 + t-s)^{-\frac{n+k_0-1}{2}} dy' ds \\
&= cr^{\alpha - \frac{n+1}{p} - 1} \int_0^t \int_{|y'| \leq 3r} s^{-\frac{1}{2}} (x_n^2 + s)^{-\frac{l_0}{2}} (|y'|^2 + x_n^2 + s)^{-\frac{n+k_0-1}{2}} dy' ds.
\end{aligned} \tag{5.16}$$

Note that

$$\begin{aligned}
\int_{|x' - y'| \leq 3r} \frac{1}{(|x' - y'|^2 + x_n^2 + s)^{\frac{n+k_0-1}{2}}} dy' &= c \int_0^{3r} \frac{\rho^{n-2}}{(\rho^2 + x_n^2 + s)^{\frac{n+k_0-1}{2}}} d\rho \\
&= c(x_n^2 + s)^{-\frac{k_0}{2}} \int_0^{\frac{3r}{\sqrt{x_n^2 + s}}} \frac{\rho^{n-2}}{(\rho^2 + 1)^{\frac{n+k_0-1}{2}}} d\rho \\
&\leq \begin{cases} c(x_n^2 + s)^{-\frac{k_0}{2}} & r \geq x_n, \\ c(x_n^2 + s)^{-\frac{k_0}{2}} \left(\frac{r}{x_n}\right)^{n-1} & r \leq x_n. \end{cases}
\end{aligned} \tag{5.17}$$

Taking $\theta_2 := \frac{k_0 + l_0}{2} \geq \frac{1}{2}$ in (5.12) - (5.13), we obtain the estimates (5.14) and (5.15). \square

Lemma 5.4. *Let $t \leq (2r)^2$ and $|x'| \leq 2r$. Let $l_0 \geq 1$.*

(1) *For $x_n^2 \leq t$ we have*

$$|D_{x_n}^{l_0} u(x, t)| \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 1} \ln(1 + \frac{r}{x_n}) \ln(1 + \frac{t}{x_n}) & \text{if } l_0 = 1, \\ cr^{\alpha - \frac{n+1}{p} - 1} x_n^{1-l_0} \ln(1 + \frac{r}{x_n}) & \text{if } l_0 \geq 2. \end{cases} \tag{5.18}$$

(2) *For $t \leq x_n^2$, then*

$$|D_{x_n}^{l_0} u(x, t)| \leq cr^{\alpha - \frac{n+1}{p} - 1} x_n^{-l_0} t^{\frac{1}{2}} \left(\chi_{x_n \leq r}(x_n) \ln(1 + \frac{r}{x_n}) + \chi_{x_n \geq r}(x_n) r x_n^{-1} \right). \tag{5.19}$$

Proof. Decompose u^i by $u^i = u_1^i + u_2^i$, where

$$u_1^i(x, t) = -2\delta_{ij} \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - y', x_n, t-s) a(y', s) dy' ds, \tag{5.20}$$

$$u_2^i(x, t) = 4 \int_0^t \int_{\mathbb{R}^{n-1}} G_{ij}(x' - y', x_n, t-s) a(y', s) dy' ds. \tag{5.21}$$

First, we estimate $D_{x_n}^{l_0} u_1^i$. Note that for $x_n > 0$, $D_{x_n}^2 \Gamma = D_t \Gamma - \Delta_{n-1} \Gamma$ in $\mathbb{R}_+^n \times (0, \infty)$, where $\Delta_{n-1} = \sum_{k=1}^{n-1} D_{x_k}^2$. Hence, if $l_0 \geq 1$, then

$$\begin{aligned} D_{x_n}^{l_0} u_1^i(x, t) &= -2\delta_{ij} \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n}^{l_0+1} \Gamma(x' - y', x_n, t-s) a(y', s) dy' ds \\ &= -2\delta_{ij} \int_0^t \int_{\mathbb{R}^{n-1}} (D_t - \Delta_{n-1}) D_{x_n}^{l_0-1} \Gamma(x' - y', x_n, t-s) a(y', s) dy' ds \\ &= -2\delta_{ij} \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n}^{l_0-1} \Gamma(x' - y', x_n, t-s) D_s a(y', s) dy' ds \\ &\quad - 2\delta_{ij} \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_k} D_{x_n}^{l_0-1} \Gamma(x' - y', x_n, t-s) D_{y_k} a(y', s) dy' ds. \end{aligned}$$

Note that for each multi-index m , there is $c(m) > 0$ such that $|D_x^m \Gamma(x, t)| \leq c(m) t^{-\frac{n+|m|}{2}} e^{-\frac{|x|^2}{2t}}$. Hence, for $t \leq (2r)^2$ and $|x'| \leq 2r$, we have

$$\begin{aligned} |D_{x_n}^{l_0} u_1^i(x, t)| &\leq cr^{\alpha - \frac{n+1}{p} - 2} \int_0^t \int_{|y'| < r} (t-s)^{-\frac{n+l_0-1}{2}} e^{-\frac{|x'-y'|^2 + x_n^2}{2(t-s)}} dy' ds \\ &\quad + cr^{\alpha - \frac{n+1}{p} - 1} \int_0^t \int_{|y'| < r} (t-s)^{-\frac{n+l_0}{2}} e^{-\frac{|x'-y'|^2 + x_n^2}{2(t-s)}} dy' ds \\ &\leq cr^{\alpha - \frac{n+1}{p} - 2} \int_0^t \int_{|y'| < 3r} s^{-\frac{n+l_0-1}{2}} e^{-\frac{|y'|^2 + x_n^2}{2s}} dy' ds \\ &\quad + cr^{\alpha - \frac{n+1}{p} - 1} \int_0^t \int_{|y'| < 3r} s^{-\frac{n+l_0}{2}} e^{-\frac{|y'|^2 + x_n^2}{2s}} dy' ds \\ &\leq cr^{\alpha - \frac{n+1}{p} - 2} \int_0^t s^{-\frac{l_0}{2}} e^{-\frac{x_n^2}{2s}} ds + cr^{\alpha - \frac{n+1}{p} - 1} \int_0^t s^{-\frac{l_0+1}{2}} e^{-\frac{x_n^2}{2s}} ds. \end{aligned} \tag{5.22}$$

Note that $e^{-\frac{x_n^2}{2s}} \leq c(\frac{x_n^2}{s} + 1)^{-\frac{m}{2}}$ for any $m \geq 0$. Hence, for $x_n^2 \leq t$ we have

$$\begin{aligned} \int_0^t s^{-\frac{l_0}{2}} e^{-\frac{x_n^2}{2s}} ds &\leq c \int_0^t s^{-\frac{1}{2}} (x_n^2 + s)^{-\frac{l_0-1}{2}} ds \leq cx_n^{-l_0+1} t^{\frac{1}{2}} \leq cx_n^{-l_0+1} r, \\ \int_0^t s^{-\frac{l_0+1}{2}} e^{-\frac{x_n^2}{2s}} ds &\leq c \int_0^t s^{-\frac{1}{2}} (x_n^2 + s)^{-\frac{l_0}{2}} ds \leq cx_n^{-l_0} \int_0^{x_n^2} s^{-\frac{1}{2}} ds + c \int_{x_n^2}^t s^{-\frac{l_0+1}{2}} ds \\ &\leq \begin{cases} c \ln(1 + \frac{t}{x_n^2}) & \text{for } l_0 = 1, \\ cx_n^{-l_0+1} & \text{for } l_0 \neq 1. \end{cases} \end{aligned}$$

For $x_n^2 \geq t$ and $a := \frac{l_0}{2}$ or $a := \frac{l_0+1}{2}$, we have

$$\int_0^t s^{-a} e^{-\frac{x_n^2}{2s}} ds = x_n^{-2a+2} \int_{\frac{x_n^2}{t}}^\infty s^{a-2} e^{-\frac{1}{2}s} ds \leq cx_n^{-2a+2} e^{-\frac{x_n^2}{4t}}.$$

Applying the above estimates to the right hand side of (5.22), we have for $x_n^2 \leq t$

$$|D_{x_n}^{l_0} u_1^i(x, t)| \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 1} \ln(1 + \frac{t}{x_n^2}) & \text{for } l_0 = 1, \\ cr^{\alpha - \frac{n+1}{p} - 1} x_n^{-l_0+1} & \text{for } l_0 \geq 2 \end{cases} \tag{5.23}$$

and for $x_n^2 \geq t$

$$|D_{x_n}^{l_0} u_1^i(x, t)| \leq cr^{\alpha - \frac{n+1}{p} - 2} x_n^{-l_0+2} e^{-\frac{x_n^2}{4t}} + r^{\alpha - \frac{n+1}{p} - 1} x_n^{-l_0+1} e^{-\frac{x_n^2}{4t}}. \quad (5.24)$$

Next, we would like to estimate $D_{x_n}^{l_0} u_2^i$. Observe that $\int_{\mathbb{R}^{n-1}} G_{ij}(y', x_n, t) dy' = 0$ for $1 \leq i \leq n$, $1 \leq j \leq n-1$. Hence,

$$D_{x_n}^{l_0} u_2^i(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n}^{l_0} G(x' - y', x_n, t-s) (a(y', s) - a(x', s)) dy' ds.$$

From the estimate (3.2) of G_{ij} , we have that for $t \leq (2r)^2$ and $|x'| \leq 2r$

$$\begin{aligned} |D_{x_n}^{l_0} u_2^i(x, t)| &\leq cr^{\alpha - \frac{n+1}{p} - 1} \int_0^t \int_{|y'| \leq 3r} \frac{|x' - y'|}{(t-s)^{\frac{1}{2}} (|x' - y'|^2 + x_n^2 + t-s)^{\frac{n}{2}} (x_n^2 + t-s)^{\frac{l_0}{2}}} dy' ds \\ &\quad + r^{\alpha - \frac{n+1}{p}} \int_0^t \int_{|y'| \geq 3r} \frac{1}{(t-s)^{\frac{1}{2}} (|x' - y'|^2 + x_n^2 + t-s)^{\frac{n}{2}} (x_n^2 + t-s)^{\frac{l_0}{2}}} dy' ds \\ &\leq cr^{\alpha - \frac{n+1}{p} - 1} \int_0^t \int_{|x' - y'| \leq 5r} \frac{|x' - y'|}{s^{\frac{1}{2}} (|x' - y'|^2 + x_n^2 + s)^{\frac{n}{2}} (x_n^2 + s)^{\frac{l_0}{2}}} dy' ds \\ &\quad + cr^{\alpha - \frac{n+1}{p}} \int_0^t \int_{|x' - y'| \geq r} \frac{1}{s^{\frac{1}{2}} (|x' - y'|^2 + x_n^2 + s)^{\frac{n}{2}} (x_n^2 + s)^{\frac{l_0}{2}}} dy' ds. \end{aligned} \quad (5.25)$$

Note that

$$\begin{aligned} \int_{|x' - y'| \leq 5r} \frac{|x' - y'|}{(|x' - y'|^2 + x_n^2 + s)^{\frac{n}{2}}} dy' &\leq c \int_0^{5r} \frac{\rho^{n-1}}{(\rho^2 + x_n^2 + s)^n} d\rho \\ &= c \int_0^{\frac{5r}{\sqrt{x_n^2 + s}}} \frac{\rho^{n-1}}{(\rho^2 + 1)^{\frac{n}{2}}} d\rho \\ &\leq c \ln(1 + \frac{r}{x_n}). \end{aligned} \quad (5.26)$$

And

$$\int_{|x' - y'| \geq r} \frac{1}{(|x' - y'|^2 + x_n^2 + s)^{\frac{n}{2}}} dy' \leq c \int_r^\infty \frac{1}{(\rho + \sqrt{x_n^2 + s})^2} d\rho \leq c(x_n + r)^{-1}. \quad (5.27)$$

Applying the (5.26) and (5.27) to the right hand side of (5.25), we have

$$|D_{x_n}^{l_0} u_2^i(x, t)| \leq cr^{\alpha - \frac{n+1}{p} - 1} \left(\chi_{x_n \leq r}(x_n) \ln(1 + \frac{r}{x_n}) + \chi_{x_n \geq r}(x_n) r x_n^{-1} \right) \int_0^t s^{-\frac{1}{2}} (x_n^2 + s)^{-\frac{l_0}{2}} ds. \quad (5.28)$$

Taking $\theta_2 = \frac{l_0}{2} \geq \frac{1}{2}$ in (5.12) and (5.13), from (5.28), we have that for $x_n^2 \leq t \leq (2r)^2$

$$|D_{x_n}^{l_0} u_2^i(x, t)| \leq \begin{cases} cr^{\alpha - \frac{n+1}{p} - 1} \ln(1 + \frac{r}{x_n}) \ln(1 + \frac{t}{x_n^2}) & \text{for } l_0 = 1, \\ cr^{\alpha - \frac{n+1}{p} - 1} \ln(1 + \frac{r}{x_n}) x_n^{1-l_0} & \text{for } l_0 \geq 2, \end{cases} \quad (5.29)$$

and for $x_n^2 \geq t$

$$|D_{x_n}^{l_0} u_2^i(x, t)| \leq cr^{\alpha - \frac{n+1}{p} - 1} x_n^{-l_0} t^{\frac{1}{2}} \left(\chi_{x_n \leq r}(x_n) (1 + \ln \frac{r}{x_n}) + \chi_{x_n \geq r}(x_n) r x_n^{-1} \right). \quad (5.30)$$

From (5.23), (5.24), (5.29) and (5.30), we obtain (5.18) and (5.19). \square

6. ESTIMATES OF SOLUTION IN $\mathbb{R}_+^n \times (0, \infty)$

Theorem 6.1. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let $g = (g', 0) \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, \infty))$ with $g(x', 0) = 0$ if $\alpha > \frac{2}{p}$. Then there is a solution u of the Stokes system (1.4) in $\mathbb{R}^{n-1} \times (0, \infty)$ with boundary data g such that*

$$\int_0^\infty \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{-\alpha p + (k_0 + l_0 + 2m_0)p - 1} |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u(x, t)|^p dx dt \leq c \|g\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0, \infty))}^p \quad (6.1)$$

for $k_0 + l_0 + 2m_0 = 1, 2, 3$.

Proof. By Proposition 4.4, it is sufficient to consider g which consists of the (α, p) atoms. For simplicity, assume $g_k = a\delta_{kj}$, $j = 1, \dots, n-1$ for some atom a supported on $\Delta(0, r^2) \times (0, r^2)$. We denote $\beta = -\alpha p + (k_0 + l_0 + 2m_0)p - 1$. Let u be defined by (5.1). Decompose the domain of integration into four parts so that

$$\int_0^\infty \int_0^\infty \int_{\mathbb{R}^{n-1}} (x_n \wedge t^{\frac{1}{2}})^\beta |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt = \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned} I_1 &= \int_0^{(2r)^2} \int_0^\infty \int_{|x'| \leq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt, \\ I_2 &= \int_{(2r)^2}^\infty \int_0^\infty \int_{|x'| \leq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt, \\ I_3 &= \int_0^{(2r)^2} \int_0^\infty \int_{|x'| \geq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt, \\ I_4 &= \int_{(2r)^2}^\infty \int_0^\infty \int_{|x'| \geq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt. \end{aligned}$$

• From the estimate (5.2)₁, we have

$$\begin{aligned} I_4 &\leq cr^{p\alpha - n - 1 + (n+1)p} \int_{(2r)^2}^\infty \int_0^\infty \int_{|x'| \geq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta \\ &\quad \times t^{-(\frac{1}{2} + m_0)p} (x_n^2 + t)^{-\frac{l_0 p}{2}} (|x'|^2 + x_n^2 + t)^{-\frac{(n+k_0)p}{2}} dx' dx_n dt \\ &\leq cr^{p\alpha - n - 1 + (n+1)p} \int_{(2r)^2}^\infty \int_0^\infty (x_n \wedge t^{\frac{1}{2}})^\beta t^{-(\frac{1}{2} + m_0)p} (x_n^2 + t)^{-\frac{(n+k_0+l_0)p}{2} + \frac{n-1}{2}} dx_n dt. \end{aligned} \quad (6.2)$$

Note that for $\theta_1 > -1$ and $\theta_2 > \frac{1}{2}$, we have

$$\begin{aligned} \int_0^\infty (x_n \wedge t^{\frac{1}{2}})^{\theta_1} (x_n^2 + t)^{-\theta_2} dx_n &\leq t^{-\theta_2} \int_0^{t^{\frac{1}{2}}} x_n^{\theta_1} dx_n + t^{\frac{\theta_1}{2}} \int_{t^{\frac{1}{2}}}^\infty x_n^{-2\theta_2} dx_n \\ &= ct^{-\theta_2 + \frac{1}{2}\theta_1 + \frac{1}{2}}. \end{aligned} \quad (6.3)$$

Hence, taking $\theta_1 := \beta > -1$ and $\theta_2 := \frac{(n+k_0+l_0)p}{2} - \frac{n-1}{2} > \frac{1}{2}$ in (6.3), from (6.2), we have

$$I_4 \leq cr^{p\alpha-n-1+(n+1)p} \int_{(2r)^2}^{\infty} t^{\frac{n-1}{2} - \frac{(n+\alpha+1)p}{2}} dt = c \quad \text{for } \alpha < k_0 + l_0 + 2m_0 \text{ and } p > 1. \quad (6.4)$$

• From the estimate (5.2)₂, we have

$$\begin{aligned} I_2 &\leq cr^{p\alpha-n-1+(n+1)p} \int_{(2r)^2}^{\infty} \int_0^{\infty} \int_{|x'| \leq 2r} (x_n \wedge t^{\frac{1}{2}})^{\beta} t^{-(\frac{1}{2}+m_0)p} (x_n^2 + t)^{-\frac{(n+k_0+l_0)p}{2}} dx' dx_n dt \\ &= cr^{p\alpha-2+(n+1)p} \int_{(2r)^2}^{\infty} \int_0^{\infty} (x_n \wedge t^{\frac{1}{2}})^{\beta} t^{-(\frac{1}{2}+m_0)p} (x_n^2 + t)^{-\frac{(n+k_0+l_0)p}{2}} dx_n dt. \end{aligned}$$

Hence, taking $\theta_1 := \beta > -1$ and $\theta_2 := \frac{n+k_0+l_0}{2} > \frac{1}{2}$ in (6.3), we have

$$I_2 \leq cr^{p\alpha-2+p(n+1)} \int_{(2r)^2}^{\infty} t^{-\frac{(n+\alpha+1)p}{2}} dt = c. \quad (6.5)$$

• For the estimate of I_1 , we divide the domain of integration so that

$$I_1 = I_{11} + I_{12}, \quad (6.6)$$

where

$$\begin{aligned} I_{11} &= \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^{\beta} |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt, \\ I_{12} &= \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^{\infty} \int_{|x'| \leq 2r} t^{\frac{1}{2}\beta} |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt. \end{aligned}$$

First we estimate I_{12} .

1). Let $m_0 \geq 1$. From the estimate (5.7) we have

$$\begin{aligned} I_{12} &\leq cr^{\alpha p-n-1-2m_0 p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^{\infty} \int_{|x'| \leq 2r} t^{\frac{1}{2}\beta} x_n^{-(k_0+l_0+1)p} t^{\frac{p}{2}} dx' dx_n dt \\ &= cr^{\alpha p-2-2m_0 p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^{\infty} t^{\frac{1}{2}\beta + \frac{p}{2}} x_n^{-(k_0+l_0+1)p} dx_n dt \\ &\leq cr^{\alpha p-2-2m_0 p} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + m_0 p} dt = c \quad \text{for } \alpha < 2m_0 + \frac{2}{p}. \end{aligned}$$

2). Let $m_0 = 0$ and $k_0 \geq 1$. From the estimate (5.15) we have

$$\begin{aligned}
I_{12} &\leq cr^{\alpha p - n - 1 - p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^r \int_{|x'| \leq 2r} t^{\frac{1}{2}\beta} x_n^{-(k_0 + l_0)p} t^{\frac{p}{2}} dx' dx_n dt \\
&\quad + cr^{\alpha p - n - 1 + (n-2)p} \int_0^{(2r)^2} \int_r^\infty \int_{|x'| \leq 2r} t^{\frac{1}{2}\beta} x_n^{-(n-1+k_0+l_0)p} t^{\frac{p}{2}} dx' dx_n dt \\
&\leq cr^{\alpha p - 2 - p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^r t^{\frac{1}{2}\beta + \frac{p}{2}} x_n^{-(k_0 + l_0)p} dx_n dt \\
&\quad + cr^{\alpha p - 2 + (n-2)p} \int_0^{(2r)^2} \int_r^\infty t^{\frac{1}{2}\beta + \frac{1}{2}p} x_n^{-(n-1+k_0+l_0)p} dx_n dt \\
&\leq cr^{\alpha p - 2 - p} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + \frac{p}{2}} dt + cr^{\alpha p - (1+k_0+l_0)p-1} \int_0^{(2r)^2} t^{\frac{1}{2}\beta + \frac{1}{2}p} dt \\
&= c \quad \text{for } \alpha < 1 + \frac{2}{p}.
\end{aligned}$$

3). Let $m_0 = k_0 = 0$ and $l_0 \geq 1$. From the estimate (5.19) we have

$$\begin{aligned}
I_{12} &\leq cr^{\alpha p - n - 1 - p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^r \int_{|x'| \leq 2r} t^{\frac{1}{2}\beta} x_n^{-l_0 p} \ln^p(1 + \frac{r}{x_n}) t^{\frac{p}{2}} dx' dx_n dt \\
&\quad + cr^{\alpha p - n - 1} \int_0^{(2r)^2} \int_r^\infty \int_{|x'| \leq 2r} t^{\frac{1}{2}\beta} x_n^{-l_0 p} x_n^{-p} t^{\frac{p}{2}} dx' dx_n dt \\
&\leq cr^{\alpha p - p - 2} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^r t^{\frac{1}{2}\beta + \frac{p}{2}} x_n^{-l_0 p} \ln^p(1 + \frac{r}{x_n}) dx_n dt \\
&\quad + cr^{\alpha p - 2} \int_0^{(2r)^2} \int_r^\infty t^{\frac{1}{2}\beta + \frac{p}{2}} x_n^{-(l_0+1)p} dx_n dt.
\end{aligned} \tag{6.7}$$

Here

$$\int_0^{(2r)^2} \int_r^\infty t^{\frac{1}{2}\beta + \frac{p}{2}} x_n^{-(l_0+1)p} dx_n dt = cr^{-(l_0+1)p+1} \int_0^{(2r)^2} t^{\frac{1}{2}\beta + \frac{p}{2}} dt = cr^{-\alpha p + 2}. \tag{6.8}$$

Note that for $0 < \epsilon < 1$ there is a positive constant $c(\epsilon) > 0$ such that for $a > 0$

$$\ln(1 + a) \leq c(\epsilon) a^\epsilon. \tag{6.9}$$

Hence, taking $\epsilon > 0$ sufficiently small in (6.9) ($\epsilon < 1 + \frac{2}{p} - \alpha$), we have

$$\begin{aligned}
\int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^r t^{\frac{1}{2}\beta + \frac{p}{2}} x_n^{-l_0 p} \ln^p(1 + \frac{r}{x_n}) dx_n dt &\leq c(\epsilon) \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^r t^{\frac{1}{2}\beta + \frac{p}{2}} x_n^{-l_0 p} (\frac{r}{x_n})^{\epsilon p} dx_n dt \\
&\leq c(\epsilon) \int_0^{(2r)^2} t^{\frac{1}{2}\beta + \frac{p}{2}} r^{\epsilon p} t^{-\frac{l_0}{2}p + \frac{1}{2} - \frac{1}{2}\epsilon p} dt \\
&= c(\epsilon) r^{-\alpha p + p + 2}.
\end{aligned} \tag{6.10}$$

From (6.7), (6.8) and (6.10), we obtain

$$I_{12} \leq c. \tag{6.11}$$

Second, we estimate I_{11} .

1). Let $m_0 \geq 1$ and $k_0 + l_0 \geq 1$. Then from the estimate (5.6)₁ we have

$$\begin{aligned} I_{11} &\leq cr^{\alpha p - n - 1 - 2m_0 p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta x_n^{-(k_0 + l_0)p} dx' dx_n dt \\ &\leq cr^{\alpha p - 2 - 2m_0 p} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + m_0 p} dt = c \text{ for } \alpha < 2m_0. \end{aligned}$$

2). Let $m_0 \geq 1$ and $k_0 = l_0 = 0$. From the estimate (5.6)₂ and taking $\epsilon > 0$ sufficiently small in (6.9) ($\frac{\epsilon}{p} < 2m_0 + \frac{1}{p} - \alpha$), we have

$$\begin{aligned} I_{11} &\leq cr^{\alpha p - n - 1 - 2m_0 p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta \ln^p(1 + \frac{t}{x_n^2}) dx' dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - 2m_0 p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^\beta (\frac{t}{x_n^2})^{\epsilon p} dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - 2m_0 p} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + m_0 p} dt = c(\epsilon). \end{aligned}$$

3). Let $m_0 = 0$ and $k_0 \geq 1, k_0 + l_0 \geq 2$. Then from the estimate (5.14)₂, we have

$$\begin{aligned} I_{11} &\leq cr^{\alpha p - n - 1 - p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta x_n^{(1 - k_0 - l_0)p} dx' dx_n dt \\ &\leq cr^{\alpha p - 2 - p} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + \frac{p}{2}} dt = c \text{ for } \alpha < 1. \end{aligned}$$

4). Let $m_0 = 0$ and $k_0 = 1$. Then from the estimate (5.14)₁ and (6.9) (taking ϵ satisfying $0 < \frac{\epsilon}{p} < 1 - \alpha$), we have

$$\begin{aligned} I_{11} &\leq cr^{\alpha p - n - 1 - p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta \ln^p(1 + \frac{t}{x_n^2}) dx' dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^\beta (\frac{t}{x_n^2})^{\epsilon p} dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - p} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + \frac{p}{2}} dt = c(\epsilon). \end{aligned}$$

5). Let $m_0 = k_0 = 0$ and $l_0 \geq 2$. Then from the estimate (5.18)₂ and (6.9) (taking ϵ satisfying $0 < \frac{\epsilon}{p} < 1 - \alpha$), we have

$$\begin{aligned} I_{11} &\leq cr^{\alpha p - n - 1 - p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta x_n^{p - l_0 p} \ln^p(1 + \frac{r}{x_n}) dx' dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^{\beta + p - l_0 p} (\frac{r}{x_n})^{\epsilon p} dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - p} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + \frac{p}{2}} r^{\epsilon p} t^{-\frac{\epsilon}{2} p} dt = c(\epsilon). \end{aligned}$$

6). Let $m_0 = k_0 = 0$ and $l_0 = 1$. Then from the estimate (5.18)₁ and (6.9) (taking ϵ satisfying $0 < \frac{\epsilon}{p} < 1 - \alpha$), we have

$$\begin{aligned} I_{11} &\leq cr^{\alpha p - n - 1 - p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta \ln^p(1 + \frac{r}{x_n}) dx' dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^\beta (\frac{r}{x_n})^{\epsilon p} dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2 - p} \int_0^{(2r)^2} t^{\frac{1}{2}\beta + \frac{1}{2}} r^{\epsilon p} t^{-\frac{1}{2}\epsilon p} dt = c(\epsilon). \end{aligned}$$

From 1) to 6), we get that $I_{11} \leq c$ and hence with (6.11) and (6.6), we get

$$I_1 \leq c. \quad (6.12)$$

- For the estimate of I_3 , we divide the domain of integration so that

$$I_3 = I_{31} + I_{32},$$

where

$$\begin{aligned} I_{31} &= \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \geq 2r} x_n^\beta |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt, \\ I_{32} &= \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^\infty \int_{|x'| \geq 2r} t^{\frac{1}{2}\beta} |D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} u|^p dx' dx_n dt. \end{aligned}$$

First, we estimate I_{32} . From the estimate (5.9) we have

$$I_{32} \leq cr^{\alpha p - n - 1 - 2m_0 p + (n-1)p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^\infty \int_{|x'| \geq 2r} t^{\frac{1}{2}\beta} x_n^{-l_0 p} (|x'|^2 + x_n^2)^{-\frac{(n+k_0)p}{2}} t^{\frac{p}{2}} dx' dx_n dt.$$

Note that for $\gamma < -\frac{n-1}{2}$ and $A \geq 0$, we have

$$\int_{|x'| \geq 2r} (|x'|^2 + A^2)^\gamma dx' \leq c \int_{2r}^\infty (\rho + A)^{2\gamma + n - 2} d\rho \leq c(A + r)^{2\gamma + n - 1}. \quad (6.13)$$

Taking $\gamma := -\frac{(n+k_0)p}{2}$ and $A := x_n$ in (6.13), we have

$$\begin{aligned} I_{32} &\leq cr^{\alpha p - (1+k_0+2m_0)p-2} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^r t^{\frac{1}{2}\beta + \frac{1}{2}p} x_n^{-l_0 p} dx_n dt \\ &\quad + cr^{\alpha p - n - 1 - 2m_0 p + (n-1)p} \int_0^{(2r)^2} \int_r^\infty t^{\frac{1}{2}\beta + \frac{1}{2}p} x_n^{-(n+k_0+l_0)p+n-1} dx_n dt \\ &\leq c. \end{aligned} \quad (6.14)$$

Now, we estimate I_{31} .

1). Let $l_0 \neq 1$. From the estimate (5.8)₂, we have

$$I_{31} \leq cr^{\alpha p - n - 1 - 2m_0 p + (n-1)p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \geq 2r} x_n^\beta (x_n^2 + t)^{\frac{1-l_0}{2}p} (|x'|^2 + x_n^2)^{\frac{-n-k_0}{2}p} dx' dx_n dt.$$

Taking $\gamma := -\frac{(n+k_0)p}{2} < -\frac{n-1}{2}$ and $A := x_n$ in (6.13), we have

$$\begin{aligned} I_{31} &\leq cr^{\alpha p - n - 1 - 2m_0 p + (n-1)p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^\beta t^{\frac{1-l_0}{2}p} (x_n + r)^{n-1-(n+k_0)p} dx_n dt \\ &\leq cr^{(\alpha-2m_0-k_0-1)p-2} \int_0^{(2r)^2} t^{\frac{(-\alpha+k_0+2m_0+1)p}{2}} dt = c. \end{aligned}$$

2). Let $l_0 = 1$. From the estimate (5.8)₁, we have

$$\begin{aligned} I_{31} &\leq cr^{(\alpha-2m_0)p-n-1+(n-1)p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \geq 2r} x_n^\beta (|x'|^2 + x_n^2)^{-\frac{n+k_0}{2}p} \ln^p(1 + \frac{t}{x_n^2}) dx' dx_n dt \\ &\leq c(\epsilon) r^{(\alpha-2m_0)p-n-1+(n-1)p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \geq 2r} x_n^\beta |x'|^{-(n+k_0)p} (\frac{t}{x_n^2})^\epsilon dx' dx_n dt \\ &\leq c(\epsilon) r^{(\alpha-2m_0-k_0-1)p-2} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^\beta (\frac{t}{x_n^2})^\epsilon dx_n dt \\ &\leq c(\epsilon) r^{(\alpha-2m_0-k_0-1)p-2} \int_0^{(2r)^2} t^{-\frac{\alpha p}{2} + \frac{(1+k_0+2m_0)p}{2}} dt = c(\epsilon). \end{aligned}$$

From 1) to 3) and (6.14), we get

$$I_3 \leq c. \quad (6.15)$$

Hence, from (6.4), (6.5), (6.14) and (6.15), we get (6.1). \square

Theorem 6.2. Let $T > 0$. Let $\beta > -1$. Suppose that $g \in \mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0, T))$. Then

$$\int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^\beta |u|^p dx' dx_n dt \leq cT^{\alpha p + \beta + 1} \|g\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0, T))}^p.$$

Proof. Without loss of generality, we assume $g \in \mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times \mathbb{R})$ (Otherwise, there is, $\tilde{g} \in \mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times \mathbb{R})$ so that $\tilde{g}|_{\mathbb{R}^{n-1} \times (0, T)} = g$ and $\|\tilde{g}\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|g\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0, T))}$). By Proposition 4.4, it is sufficient to consider g which consists of the (α, p) atoms (Note that we can even take atoms of g which are supported in $\mathbb{R}^{n-1} \times (-T, 2T)$). For simplicity,

assume $g_k = a\delta_{kj}, j = 1, \dots, n-1$ for some atom a supported on $\Delta(0, r) \times (0, r^2) \subset B_T$. Let u be represented by (5.1). Decompose the domain of integration into four parts so that

$$\int_0^T \int_0^\infty \int_{\mathbb{R}^{n-1}} (x_n \wedge t^{\frac{1}{2}})^\beta |u|^p dx' dx_n dt = \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned} I_1 &= \int_0^{(2r)^2} \int_0^\infty \int_{|x'| \leq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |u|^p dx' dx_n dt, \\ I_2 &= \int_{(2r)^2}^T \int_0^\infty \int_{|x'| \leq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |u|^p dx' dx_n dt, \\ I_3 &= \int_0^{(2r)^2} \int_0^\infty \int_{|x'| \geq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |u|^p dx' dx_n dt, \\ I_4 &= \int_{(2r)^2}^T \int_0^\infty \int_{|x'| \geq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta |u|^p dx' dx_n dt. \end{aligned}$$

- From the estimate (5.2)₁ and (6.13) (taking $\gamma := -\frac{np}{2}$ and $A := \sqrt{x_n^2 + t}$), we have

$$\begin{aligned} I_4 &\leq cr^{p\alpha-n-1+(n+1)p} \int_{(2r)^2}^T \int_0^\infty \int_{|x'| \geq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta t^{-\frac{1}{2}p} (|x'|^2 + x_n^2 + t)^{-\frac{np}{2}} dx' dx_n dt \\ &\leq cr^{p\alpha-n-1+p(n+1)} \int_{(2r)^2}^T \int_0^\infty (x_n \wedge t^{\frac{1}{2}})^\beta t^{-\frac{1}{2}p} (x_n^2 + t + r^2)^{-\frac{np}{2} + \frac{n-1}{2}} dx_n dt. \end{aligned}$$

Taking $\theta_1 := \beta > -1$ and $\theta_2 := \frac{np}{2} - \frac{n-1}{2} > \frac{1}{2}$ in (6.3), we have

$$\begin{aligned} I_4 &\leq cr^{p\alpha-n-1+p(n+1)} \int_{(2r)^2}^T t^{\frac{n+\beta-np}{2}} t^{-\frac{1}{2}p} dt \\ &= \begin{cases} cr^{p\alpha+1+\beta} & \text{for } \beta < (n+1)p - n - 2 \\ cr^{p\alpha-n-1+p(n+1)} T^{\frac{n-(n+1)p+\beta}{2}+1} & \text{for } \beta > (n+1)p - n - 2 \\ cr^{p\alpha-n-1+p(n+1)} \ln \frac{T}{(2r)^2} & \text{for } \beta = (n+1)p - n - 2 \end{cases} \leq cT^{\frac{p\alpha+1+\beta}{2}}. \end{aligned}$$

- From the estimate (5.2)₂, we have

$$\begin{aligned} I_2 &\leq cr^{p\alpha-n-1+p(n+1)} \int_{(2r)^2}^T \int_0^\infty \int_{|x'| \leq 2r} (x_n \wedge t^{\frac{1}{2}})^\beta t^{-\frac{p}{2}} (x_n^2 + t)^{-\frac{np}{2}} dx' dx_n dt \\ &= cr^{p\alpha-2+p(n+1)} \int_{(2r)^2}^T \int_0^\infty (x_n \wedge t^{\frac{1}{2}})^\beta t^{-\frac{p}{2}} (x_n^2 + t)^{-\frac{np}{2}} dx_n dt. \end{aligned}$$

Taking $\theta_1 := \beta > -1$ and $\theta_2 := \frac{np}{2} > \frac{1}{2}$ in (6.3), we have

$$\begin{aligned} I_2 &\leq cr^{p\alpha-2+p(n+1)} \int_{(2r)^2}^T t^{\frac{\beta-(n+1)p+1}{2}} dt \\ &= \begin{cases} cr^{p\alpha+1+\beta} & \text{for } \beta < (n+1)p - 3 \\ cr^{p\alpha-2+p(n+1)} T^{\frac{3-(n+1)p+\beta}{2}+1} & \text{for } \beta > (n+1)p - 3 \\ cr^{p\alpha-2+p(n+1)} \ln \frac{T}{(2r)^2} & \text{for } \beta = (n+1)p - 3 \end{cases} \leq cT^{\frac{p\alpha+1+\beta}{2}}. \end{aligned}$$

- For the estimate of I_1 , we divide the domain of integration so that

$$I_1 = I_{11} + I_{12},$$

where

$$\begin{aligned} I_{11} &= \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta |u|^p dx' dx_n dt, \\ I_{12} &= \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^\infty \int_{|x'| \leq 2r} t^{\frac{\beta}{2}} |u|^p dx' dx_n dt. \end{aligned}$$

From the estimate (5.7) we have

$$\begin{aligned} I_{12} &\leq cr^{\alpha p - n - 1} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^\infty \int_{|x'| \leq 2r} t^{\frac{\beta}{2}} x_n^{-p} t^{\frac{p}{2}} dx' dx_n dt \\ &\leq cr^{\alpha p - 2} \int_0^{(2r)^2} t^{\frac{\beta+1}{2}} dt = cr^{\alpha p + \beta + 1} \leq cT^{\frac{p\alpha+1+\beta}{2}}. \end{aligned}$$

From the estimate (5.6)₂ and (6.9), we have

$$\begin{aligned} I_{11} &\leq cr^{\alpha p - n - 1} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \leq 2r} x_n^\beta \ln^p(1 + \frac{t}{x_n^2}) dx' dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^\beta (\frac{t}{x_n^2})^{\epsilon p} dx_n dt \\ &\leq c(\epsilon) r^{\alpha p - 2} \int_0^{(2r)^2} t^{\frac{\beta+1}{2}} dt = c(\epsilon) r^{\alpha p + \beta + 1} \leq c(\epsilon) T^{\frac{p\alpha+1+\beta}{2}}. \end{aligned}$$

- For the estimate of I_3 , we divide the domain of integration so that

$$I_3 = I_{31} + I_{32},$$

where

$$\begin{aligned} I_{31} &= \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \geq 2r} x_n^\beta |u|^p dx' dx_n dt \\ I_{32} &= \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^\infty \int_{|x'| \geq 2r} t^{\frac{1}{2}\beta} |u|^p dx' dx_n dt. \end{aligned}$$

From the estimate (5.9) we have

$$I_{32} \leq cr^{\alpha p - n - 1 + (n-1)p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^\infty \int_{|x'| \geq 2r} t^{\frac{\beta}{2}} (|x'|^2 + x_n^2)^{-\frac{np}{2}} t^{\frac{p}{2}} dx' dx_n dt.$$

Taking $\gamma := -\frac{np}{2} < -\frac{n-1}{2}$ and $A := x_n$ in (6.13), we have

$$\begin{aligned} I_{32} &\leq cr^{\alpha p - n - 1 + (n-1)p} \int_0^{(2r)^2} \int_{t^{\frac{1}{2}}}^\infty t^{\frac{\beta}{2}} (x_n^2 + r^2)^{-\frac{np}{2} + \frac{n-1}{2}} t^{\frac{p}{2}} dx_n dt \\ &\leq cr^{\alpha p - 1 - p} \int_0^{(2r)^2} t^{\frac{\beta+p}{2}} dt = cr^{p\alpha+1+\beta} \leq cT^{\frac{p\alpha+1+\beta}{2}}. \end{aligned}$$

From the estimate (5.8)₂ we have

$$I_{31} \leq cr^{\alpha p - n - 1 + (n-1)p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} \int_{|x'| \geq 2r} x_n^{\beta+p} (|x'| + x_n)^{-np} dx' dx_n dt.$$

Taking $\gamma := -\frac{np}{2} < -\frac{n-1}{2}$ and $A := x_n$ in (6.13), we have

$$\begin{aligned} I_{31} &\leq cr^{\alpha p - n - 1 + (n-1)p} \int_0^{(2r)^2} \int_0^{t^{\frac{1}{2}}} x_n^{\beta+p} (x_n + r)^{-np+n-1} dx_n dt \\ &\leq cr^{(\alpha-1)p-2} \int_0^{(2r)^2} t^{\frac{\beta+p+1}{2}} dt = cr^{\alpha p + \beta + 1} \leq cT^{\frac{p\alpha+1+\beta}{2}}. \end{aligned}$$

□

7. PROOF OF THEOREM 1.6

Theorem 1.1 will be obtained by combining Theorem 7.1 below and Proposition 3.1.

Theorem 7.1. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let $g = (g', 0) \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, \infty))$. Let $T > 0$. Then there is a solution u of the Stokes system (1.4) in $\mathbb{R}^{n-1} \times (0, \infty)$ with boundary data g such that*

$$\|u\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}_+^n \times (0, T))} \leq c(T) \|g\|_{\mathcal{B}_p^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^{n-1} \times (0, T))}. \quad (7.16)$$

Proof. Let $T > 0$ and $1 < p < \infty$. Let $g \in \mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))$. Let $0 < \alpha < 1 - \frac{1}{p}$. From Lemma 4.1 we have the following estimate.

$$\begin{aligned} \|u\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}_+^n \times (0, T))}^p &\leq \int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{p-p\alpha-1} |D_x u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |D_t u|^p \\ &\quad + (x_n \wedge t^{\frac{1}{2}})^{p-p\alpha-1} |u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |u|^p dx dt. \end{aligned}$$

By Theorem 6.1, we have

$$\int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{p-p\alpha-1} |D_x u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |D_t u|^p \leq c \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))}^p.$$

By Theorem 6.2, we have

$$\int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{p-p\alpha-1} |u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |u|^p dx dt \leq c(T^{\frac{p}{2}} + T^p) \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))}^p.$$

Therefore, we conclude that for $0 < \alpha < 1 - \frac{1}{p}$

$$\|u\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}_+^n \times (0, T))} \leq C(T) \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))}.$$

Let $1 - \frac{1}{p} < \alpha < 1$. From Lemma 4.2, we have the following estimate.

$$\begin{aligned} & \|u\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}_+^n \times (0, T))}^p \\ & \leq c \int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} \left(|D_x^2 u|^p + |D_t u|^p + (x_n \wedge t^{\frac{1}{2}})^p |D_x D_t u|^p \right) \\ & \quad + (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha-1} |D_t u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |D_x u|^p \\ & \quad + (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha-1} |D_x u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |u|^p dx dt. \end{aligned}$$

By Theorem 6.1, we have

$$\int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} \left(|D_x^2 u|^p + |D_t u|^p + (x_n \wedge t^{\frac{1}{2}})^p |D_x D_t u|^p \right) \leq c \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))}^p.$$

Note that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha-1} |D_t u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |D_x u|^p + (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha-1} |D_x u|^p dx dt \\ & \leq \int_0^T \int_{\mathbb{R}_+^n} T^p (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |D_t u|^p + (T^p + T^{2p}) (x_n \wedge t^{\frac{1}{2}})^{p-p\alpha-1} |D_x u|^p dx dt. \end{aligned}$$

Hence by Theorem 6.1, we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha-1} |D_t u|^p + (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |D_x u|^p + (x_n \wedge t^{\frac{1}{2}})^{3p-p\alpha-1} |D_x u|^p dx dt \\ & \leq c(T^p + T^{2p}) \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))}^p. \end{aligned}$$

From Theorem 6.2, we have

$$\int_0^T \int_{\mathbb{R}_+^n} (x_n \wedge t^{\frac{1}{2}})^{2p-p\alpha-1} |u|^p dx dt \leq CT^{2p} \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))}^p.$$

Therefore, we conclude that for $1 - \frac{1}{p} < \alpha < 1$

$$\|u\|_{\mathcal{B}_p^{\alpha+\frac{1}{p}, \frac{1}{2}\alpha+\frac{1}{2p}}(\mathbb{R}_+^n \times (0, T))} \leq C(T) \|g\|_{\mathcal{B}_p^{\alpha, \frac{1}{2}\alpha}(\mathbb{R}^{n-1} \times (0, T))}.$$

For $\alpha = 1 - \frac{1}{p}$, we use the real interpolation. □

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